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Twisting κ-deformed phase space

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Abstract

We briefly discuss the twisting procedure applied to the κ-deformed spacetime. It appears that one can consider only two kinds of such twistings: in space and time directions. For both types of twistings we introduce related phase spaces and consider briefly their properties. We discuss in detail the changes of duality relations under the action of twist. The Jordanian twisted spacetime and phase space in $D = 2$ are also commented upon.

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1. Introduction

Recently, some suggestions appeared that the classical Lorentz invariance should be treated as an approximate symmetry in ultra-high energy processes (shift of the Greisen–Zatsepin–Kuzmin (GZK) kinematic threshold) and the relativistic spacetime symmetries on the Planck scale should be modified [1]. There are also some theoretical predictions coming from string theory and quantum gravity models that spacetime at very short distances of the order of the Planck length should be quantum, i.e. noncommutative [2]. One can modify the standard Lorentz or Poincaré relativistic symmetry in two different ways: obtaining a commutative spacetime (as in the standard relativistic theory) or noncommutative spacetime with space and time non-commuting variables.

The first type of modified relativistic symmetries follows from the concept of the double special relativistic (DSR) theory introduced by Amelino-Camelia [3], where two observer-independent parameters (scales)—c, λ_p—the velocity of light and Planck length play the fundamental role. In this framework, two basic models of such DSR theory are considered: DSR1 theory proposed by Amelino-Camelia and DSR2 theory considered by Magueijo and Smolin [4]. In both proposals the energy–momentum vector space is extended to the four-momentum algebra considered as the enveloping algebra of this linear energy–momentum space. In this algebra the nonlinear transformation of the basis vectors (of energy–momentum space) is realized according to DSR1 or DSR2 theory models. In this way we get four non-linearly transformed generators belonging to the energy–momentum algebra which are

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assumed to represent physical energy and momentum. Using this transformed four-momentum basis we obtain, for instance, the standard dispersion relation in a modified form which is, however, quite equivalent (under an inverse nonlinear transformation) to the standard one. One can also consider the action of the relativistic symmetry on physical energy and momentum as a nonlinear realization of the Lorentz group and express the addition laws of energy and momentum by the coproduct (non-linearly transformed trivial coproduct) [5] and extending in this way the energy–momentum algebra to a bialgebra structure. Spacetime related to this bialgebra structure is defined as a dual bialgebra where multiplication of the spacetime variables follows from the coproduct. The momentum coproduct is symmetric, therefore spacetime is commuting. In consistency with conclusions in [5], recently it has been observed [6] that DSR theories with symmetric coproduct law describe just the standard special relativity framework in nonlinear disguise.

The second type of modified relativistic symmetries is based on the concept of the Hopf algebra (quantum group) [7, 8] and quantum deformations of the classical Lorentz or Poincaré symmetry [9]. In this approach there is a distinguished deformation, the so called κ-deformation of the Poincaré symmetry [10], where $O(3)$-rotational symmetry is not deformed. The Hopf algebra structure of this deformation will be discussed in detail in section 2. However, we would like to stress that the four-momentum algebra for DSR1 theory and the κ-deformed theory (in bicrossproduct basis) is the same, and give us the same physical predictions. The main difference between both theories lies in their coalgebra sectors, because in the κ-deformed Poincaré algebra the coproduct of the four-momentum is non-symmetric. This fact has important consequences for the possibility of physical interpretation of the κ-deformed addition law of energy–momentum. On the other hand, this non-symmetricity of coproduct allows us to obtain non-commutativity of spacetime in the Hopf algebra framework. The idea of a non-commutativity of spacetime appeared in physics since Snyder's articles [11], where the spacetime is isomorphic to a quotient of de Sitter $SO(4, 1)$ and Lorentz $SO(3, 1)$ groups.

Further, we shall consider κ-deformed spacetime defined as a dual Hopf algebra to the κ-deformed four-momentum algebra considered as a subalgebra of κ-Poincaré Hopf algebra, i.e. the spacetime is understood in the Majid and Ruegg sense [12]. Such a spacetime has the Hopf algebra structure given by coproduct of trivial form. In this paper, we shall consider a twisting operation acting on the κ-deformed spacetime algebra. The notion of twisting of the Hopf algebra was introduced by Drinfeld [13] and then applied to enveloping algebras of simple Lie algebras [14] and the applications to physical symmetry algebras and one can it find in [15] and recently in κ-deformations of Weyl and conformal symmetries [16].

For a given pair of spacetime and four-momentum algebras one can define a phase-space algebra using the notion of cross-product algebra [9]. It is well known that although both paired algebras have the Hopf algebra structure, the phase-space algebra is no longer the Hopf algebra. The case of κ-deformed phase space is considered in [17, 18].

In section 2 we recall the Hopf algebra structure of κ-Poincaré algebra in a very useful bicrossproduct basis [12] in which the Lorentz algebra sector has a classical form. We also discuss the κ-deformed spacetime underlying the role of duality relations.

In section 3.1 we show that one can define only two kinds of twisting operations on κ-deformed spacetime: twisting in space directions (SD) and in the time direction (TD). For both cases we derive their Hopf algebra structure. In section 3.2, we extend our dual pair to a phase-space algebra and we find phase-space commutation relations. It appears that the SD-twisted phase space cannot be considered as a linear vector space because of nonlinear phase-space commutation relations.

Section 4 is devoted to the derivation of duality relations between the twisted spacetime algebra and the momentum algebra for simplicity in two-dimensional spacetime. It is more a computational part of the paper where we find that the spacetime twisting effectively changes only duality relations. In this way the basis of twisted spacetime algebra becomes non-orthogonal (one can say 'twisted') with respect to the momentum basis. We also consider $D = 2$ twisted κ-deformed spacetime by non-symmetric twisting function (Jordanian twist) [19, 20] and dual momentum algebra. The phase space for this Jordanian deformation is obtained and considered.

2. κ-Poincaré algebra in the bicrossproduct basis and κ-deformed spacetime

Let us recall the structure of κ-Poincaré algebra given in the bicrossproduct basis [12]

(i) non-deformed (*classical*) Lorentz algebra ($g_{\mu\nu} = (1, -1, -1, -1)$)

$$[M_{\mu\nu}, M_{\rho\tau}] = -\mathrm{i}(g_{\mu\rho}M_{\nu\tau} + g_{\nu\tau}M_{\mu\rho} - g_{\mu\tau}M_{\nu\rho} - g_{\nu\rho}M_{\mu\tau}) \tag{1}$$

(ii) deformed covariance relations $\left(M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, N_i = M_{i0}\right)$

$$\begin{aligned}
&[M_i, p_j] = \mathrm{i}\epsilon_{ijk}p_k \qquad [M_i, p_0] = 0\\
&[N_i, p_0] = \mathrm{i}p_i \qquad [p_\mu, p_\nu] = 0\\
&[N_i, p_j] = \mathrm{i}\delta_{ij}\left[\kappa c \sinh\left(\frac{p_0}{\kappa c}\right)\mathrm{e}^{-p_0/\kappa c} + \frac{1}{2\kappa c}(\vec{p})^2\right] - \frac{\mathrm{i}}{\kappa c}p_i p_j
\end{aligned} \tag{2}$$

where κ is the massive deformation parameter and c is the light velocity.

One can extend this algebra to the Hopf algebra defining the coalgebra sector by *coproduct* $\Delta(X)$

$$\begin{aligned}
&\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i\\
&\Delta(N_i) = N_i \otimes 1 + \mathrm{e}^{-p_0/\kappa c} \otimes N_i + \frac{1}{\kappa c}\epsilon_{ijk}p_j \otimes M_k\\
&\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0\\
&\Delta(p_i) = p_i \otimes 1 + \mathrm{e}^{-p_0/\kappa c} \otimes p_i
\end{aligned} \tag{3}$$

with *antipode* $S(X)$ and *counit* $\epsilon(X)$

$$\begin{aligned}
&S(M_i) = -M_i \qquad S(N_i) = -\mathrm{e}^{p_0/\kappa c}N_i + \frac{1}{\kappa c}\epsilon_{ijk}\,\mathrm{e}^{p_0/\kappa c}p_j M_k\\
&S(p_0) = -p_0 \qquad S(p_i) = -p_i\,\mathrm{e}^{p_0/\kappa c}
\end{aligned} \tag{4}$$

$$\epsilon(X) = 0 \qquad \epsilon(1) = 1 \qquad \text{for} \quad X = M_i, N_i, p_\mu. \tag{5}$$

Therefore, in the κ-Poincaré Hopf algebra one can distinguish the following three subalgebras

(i) classical Lorentz algebra $\mathcal{L} = \{\mathcal{M}_{\mu\nu}\}$ given by (1),

(ii) non-deformed $O(3)$-rotation algebra $\mathcal{M} = \{M_i, \Delta, S, \epsilon\}$ as the Hopf subalgebra with trivial coproduct $\Delta(M_i)$,

(iii) Abelian four-momentum algebra $\mathcal{P}_\kappa = \{p_\mu, \Delta, S, \epsilon\}$ as the Hopf subalgebra with non-symmetric coproduct $\Delta(p_\mu)$ given by the relations

$$\begin{aligned}
&[p_\mu, p_\nu] = 0 \qquad && \epsilon(p_\mu) = 0\\
&\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0 \qquad && \Delta(p_i) = p_i \otimes 1 + \mathrm{e}^{-p_0/\kappa c} \otimes p_i\\
&S(p_0) = -p_0 \qquad && S(p_i) = -p_i\,\mathrm{e}^{p_0/\kappa c}.
\end{aligned} \tag{6}$$

The Hopf four-momentum algebra $\mathcal{P}_\kappa$ can be considered as a deformation of the universal enveloping algebra of classical translation algebra, generated (in the Drinfeld sense [8]) by polynomial functions of linear momentum generators p_μ (translations).

To define the κ-deformed spacetime, we adopt the Majid and Ruegg point of view [12], namely using this algebra $\mathcal{P}_\kappa$, we define κ-deformed spacetime algebra $\mathcal{X}_\kappa$ in a natural way, as a dual algebra to the four-momentum translation algebra $\mathcal{P}_\kappa$. One can assume the standard duality relations between linear bases of both algebras, respecting the Hopf algebra structure.

Let $\mathcal{X}_\kappa$ be generated by spacetime variables $\{x_0, \vec{x}\}$ dual to four-momentum p_μ satisfying the following duality relations:

$$\langle p_\mu, x_\nu\rangle = \mathrm{i}g_{\mu\nu} \qquad g_{\mu\nu} = (1, -1, -1, -1) \tag{7}$$

$$\langle p_\mu, 1\rangle = \langle 1, x_\mu\rangle = 0 \qquad \langle 1, 1\rangle = 1 \tag{8}$$

$$\langle pq, x\rangle = \langle p \otimes q, \Delta(x)\rangle \qquad \langle p, xy\rangle = \langle \Delta(p), x \otimes y\rangle \tag{9}$$

where $p_\mu, p, q \in \mathcal{P}_\kappa$ and $x_\nu, x, y \in \mathcal{X}_\kappa$, or using the Sweedler coproduct notation (see for instance [9]) $\Delta(a) = a_{(1)} \otimes a_{(2)}$ relations (9) can be rewritten as

$$\langle pq, x\rangle = \langle p \otimes q, \Delta(x)\rangle = \langle p, x_{(1)}\rangle\langle q, x_{(2)}\rangle \tag{10}$$

$$\langle p, xy\rangle = \langle \Delta(p), x \otimes y\rangle = \langle p_{(1)}, x\rangle\langle p_{(2)}, y\rangle. \tag{11}$$

Relations (6)–(9) do not describe the Hopf algebra structure of spacetime algebra uniquely, in particular they describe a wide class of coproducts $\Delta(x_\nu)$. The form of spacetime coproduct depends on the choice of higher order duality relations and vice versa. If we assume orthogonality of spacetime and four-momentum monomial basis in the form $(i, j = 1, 2, 3,\ k, l, m, n = 0, 1, 2, \ldots)$

$$\left\langle p_i^m p_0^n, x_j^k x_0^l\right\rangle = m!n!\delta_{mk}\delta_{nl}\delta_{ij}\langle p_0, x_0\rangle^n\langle p_i, x_i\rangle^m \tag{12}$$

then we get the following form of the non-commuting spacetime Hopf algebra [12]

$$[x_0, x_i] = \left\langle \mathrm{e}^{-p_0/\kappa c}, x_0\right\rangle x_i = -\frac{\mathrm{i}}{\kappa c}x_i \qquad [x_i, x_j] = 0 \tag{13}$$

$$\Delta_0(x_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu \qquad S(x_\mu) = -x_\mu \qquad \epsilon(x_\mu) = 0 \tag{14}$$

with trivial, symmetric coproduct $\Delta_0(x_\mu)$.

In this place we would like to stress the difference between the notion of spacetime algebra and the concept of deformed (quantum or κ-deformed) spacetime—the linear span of spacetime variables x_μ. For the commuting (classical) spacetime both notions are equivalent—commuting spacetime algebra is simply the algebra of commuting (polynomial) functions on spacetime. Classical spacetime can be regarded as a dual to the four-momentum (translation) space because of the trivial momentum coproduct. However, in the deformed (quantum) case the momentum coproduct (6) contains the exponential factor which belongs to the momentum algebra. It is the reason why we use the notion of spacetime algebra for non-commuting spacetime.

Because the spacetime variables x_μ do not commute among themselves, one can also choose a spacetime monomial basis with opposite ordering and satisfying the duality relations [17]

$$\begin{aligned} \left\langle p_i^m p_0^n, x_0^l x_j^k\right\rangle &= \frac{l!\delta_{km}\delta_{ij}}{(l-n)!}\langle p_i, x_i\rangle^m\langle p_0, x_0\rangle^n\left\langle \mathrm{e}^{-mp_0/\kappa c}, x_0^{l-n}\right\rangle \\ \left\langle p_i^m p_0^n, x_0^l x_j^k\right\rangle &= 0 \qquad \text{for} \quad n > l \end{aligned} \tag{15}$$

therefore, for this ordering, we obtain non-orthogonal duality relations for $0 \leqslant n \leqslant l$. For both relations (12) and (15), one can rewrite in a more convenient form using the exponential generating function

$$\left\langle p_i^m \,\mathrm{e}^{\xi p_0}, x_j^k x_0^l \right\rangle = \left\langle \mathrm{e}^{\xi p_0}, x_0^l \right\rangle \left\langle p_i^m, x_j^k \right\rangle \tag{16}$$

$$\begin{aligned}\left\langle p_i^m \,\mathrm{e}^{\xi p_0}, x_0^l x_j^k \right\rangle &= \left\langle \mathrm{e}^{\xi p_0 - m p_0/\kappa}, x_0^l \right\rangle \left\langle p_i^m, x_j^k \right\rangle \\ &= \left\langle \mathrm{e}^{\xi p_0}, \left(x_0 - \mathrm{i}\frac{k}{\kappa c}\right)^l \right\rangle \left\langle p_i^m, x_j^k \right\rangle.\end{aligned} \tag{17}$$

Comparing these formulae one can easily obtain the general form of the spacetime commutation relations

$$\left[x_i^k, x_0^l\right] = x_i^k \left\{ x_0^l - \left(x_0 - \mathrm{i}\frac{k}{\kappa c}\right)^l \right\} \tag{18}$$

in the right-time-ordered basis (12) (with all powers of the time variable x_0 on the right).

We would like to note that the duality relations (12) have the same form as in the case of the classical Poincaré algebra with the trivial Hopf structure and its commuting dual spacetime algebra. This duality relation can be rewritten in an equivalent form (see [12])

$$\langle f(p_i, p_0), : \phi(x_j, x_0) : \rangle = \left(f\left(-\mathrm{i}\frac{\partial}{\partial x_i}, \mathrm{i}\frac{\partial}{\partial x_0}\right) \phi \right)(0, 0) \tag{19}$$

for polynomial functions f, ϕ, and $: \phi :$ denotes a right-time-ordered polynomial.

Further, we shall consider the coproducts of spacetime variables related to $\Delta_0(x_\mu)$ by a twisting procedure, and we shall discuss twisted duality relations.

3. Twisted κ-deformed spacetime and phase space

3.1. Twisting procedure

It is well known that κ-deformed spacetime (13) can be extended to the Hopf algebra up to a similarity transformation (twisting) in the coalgebra sector. The choice of coproduct in the form (14) is the simplest one; however, one can also consider a more general class of the twisted coproducts [14, 15] given by the following similarity transformation:

$$\Delta^F(x_\mu) = F \Delta_0(x_\mu) F^{-1} \tag{20}$$

where $(a \otimes b)(c \otimes d) = ac \otimes bd$ and an invertible twisting function $F \in \mathcal{X}_\kappa \otimes \mathcal{X}_\kappa$ satisfies the additional Hopf structure requirements which follow from the properties of twisted coproduct $\Delta^F(x_\mu)$

(i) *coassociativity*

$$(\Delta^F \otimes 1)\Delta^F(x_\mu) = (1 \otimes \Delta^F)\Delta^F(x_\mu) \tag{21}$$

(ii) *consistency relations*

$$(\epsilon \otimes 1) \circ \Delta^F(x_\mu) = (1 \otimes \epsilon) \circ \Delta^F(x_\mu) = x_\mu \tag{22}$$

$$(S^F \otimes 1) \circ \Delta^F(x_\mu) = (1 \otimes S^F) \circ \Delta^F(x_\mu) = 0 \tag{23}$$

where we denote $(a \otimes b) \circ (c \otimes d) = acbd$. We would like to note that twisted spacetime is defined by twisted coproduct Δ^F and antipode S^F.

The coassociativity condition (21) can be rewritten in a more familiar form as a 2-cocycle condition imposed on the twisting function F [9]

$$(1 \otimes F)(1 \otimes \Delta_0)F = (F \otimes 1)(\Delta_0 \otimes 1)F. \tag{24}$$

Without loss of generality, we assume the following exponential form [14] of the twisting function

$$F = \exp\left(\sum \phi_n \otimes \phi^n\right) \tag{25}$$

where $\phi_n, \phi^n \in \mathcal{X}_\kappa$. Then one can express the coassociativity condition (24) by the formula

$$\mathrm{e}^{1\otimes\phi_n\otimes\phi^n}\,\mathrm{e}^{\phi_n\otimes\Delta_0(\phi^n)} = \mathrm{e}^{\phi_n\otimes\phi^n\otimes 1}\,\mathrm{e}^{\Delta_0(\phi_n)\otimes\phi^n}. \tag{26}$$

Let us note that relations (13), (14) define noncommuting spacetime as a four-dimensional Lie algebra and κ-deformed spacetime algebra as an enveloping Lie algebra $\mathcal{X}_\kappa$. Therefore, one can use the results of [14] to find the twisted spacetime as a twisted Lie algebra with the Hopf structure. It is equivalent to putting the additional requirements on the twisting function F

$$(\Delta_0 \otimes 1)F = F_{13}F_{23} = F_{23}F_{13} \qquad [F_{12}, F_{23}] = 0 \tag{27}$$

where we use the standard notation

$$F_{12} = F \otimes 1 \qquad F_{23} = 1 \otimes F \qquad F_{13} = \mathrm{e}^{\phi_n\otimes 1\otimes\phi^n}. \tag{28}$$

In this framework, the elements ϕ_n, ϕ^n belong to the commutative subalgebra of $\mathcal{X}_\kappa$, with trivial coproduct

$$\begin{aligned} &\Delta_0(\phi^n) = \phi^n \otimes 1 + 1 \otimes \phi^n \qquad [\phi_n \otimes \phi^n, \Delta_0(\phi^m)] = 0 \\ &\Delta_0(\phi_n) = \phi_n \otimes 1 + 1 \otimes \phi_n \qquad [\phi_n \otimes \phi^n, \Delta_0(\phi_m)] = 0. \end{aligned} \tag{29}$$

Of course, it is not the most general case of a twisting function F. For instance, if we do not assume commutativity of F_{12}, F_{23} in (27) (so, we give up the Lie algebra twisting framework) we get

$$\Delta_0(\phi_n) = \phi_n \otimes 1 + 1 \otimes \phi_n \qquad [\phi_m, \phi_n] = 0 \tag{30}$$

$$\Delta^F(\phi^n) = \phi^n \otimes 1 + 1 \otimes \phi^n \qquad [\phi^m, \phi^n] = 0. \tag{31}$$

In this case we obtain a more general twisting function F of noncommuting spacetime (the so-called Jordanian twist, discussed in section 4).

If we consider the twisting of κ-deformed spacetime algebra, we deal with symmetric function F because the dual four-momentum algebra is a commuting one. Therefore, a symmetric twisting function F satisfying the relations (24) and (27) is given by

$$F = \mathrm{e}^{\phi\otimes\phi} = \exp[(ax_0 + b_i x_i) \otimes (ax_0 + b_j x_j)] \tag{32}$$

where the four twisting parameters a, b_i are in general the complex numbers. For this twisting function F, we immediately obtain the following formulae for the twisted coproduct of spacetime variables

$$\Delta^F(x_0) = \Delta_0(x_0) + \frac{b_i}{a}[x_i \otimes (1 - \mathrm{e}^{a\lambda\phi}) + (1 - \mathrm{e}^{a\lambda\phi}) \otimes x_i] \tag{33}$$

$$\Delta^F(x_i) = \Delta_0(x_i) + x_i \otimes (\mathrm{e}^{a\lambda\phi} - 1) + (\mathrm{e}^{a\lambda\phi} - 1) \otimes x_i \tag{34}$$

where $\lambda = -\mathrm{i}/\kappa c$ (13).

Because a time variable should not depend on the space inversion, therefore it is physically reasonable to assume that the twisting function F is a space-inversion invariant. Using this fact one can construct two twisting functions related to two Abelian subalgebras of κ-deformed spacetime. The first one is generated by the time variable x_0 and the other one by the space variables x_i (13), both with trivial coproduct (14)

twisting of space directions (SD)

$$F_0(a) = \mathrm{e}^{a x_0 \otimes x_0} \tag{35}$$

twisting of time direction (TD)

$$F(b) = \mathrm{e}^{b_{ij} x_i \otimes x_j} \tag{36}$$

where $a, b_{ij} = b_i b_j \in \mathbb{C}$ in the general case. If we consider the spacetime Hopf algebra with involution * satisfying $(a \otimes b)^* = a^* \otimes b^*$ then, the assumption of Hermiticity of spacetime generators $x_\mu^* = x_\mu$ implies the unitarity of the twisting functions, i.e. $a, b_{ij} \in i\mathbb{R}$ are pure imaginary complex numbers.

For the twisting function $F_0(a)$, using formulae (20) and (23), we obtain the SD-twisted spacetime Hopf algebra $\mathcal{X}_\kappa(\alpha)$ Hermitian basis)

$$\begin{aligned}
&[x_0, x_i] = -\frac{\mathrm{i}}{\kappa c} x_i \qquad [x_i, x_j] = 0 \\
&\Delta^{F_0(a)}(x_0) = \Delta_\alpha(x_0) = x_0 \otimes 1 + 1 \otimes x_0 \\
&\Delta^{F_0(a)}(x_i) = \Delta_\alpha(x_i) = x_i \otimes \mathrm{e}^{\alpha x_0} + \mathrm{e}^{\alpha x_0} \otimes x_i \\
&S^{F_0(a)}(x_i) = S_\alpha(x_i) = -x_i\, \mathrm{e}^{\alpha(\mathrm{i}/\kappa c - 2x_0)} \\
&S^{F_0(a)}(x_0) = S_\alpha(x_0) = -x_0 \qquad \epsilon(x_\mu) = 0
\end{aligned} \tag{37}$$

where

$$\alpha \equiv a \left\langle \mathrm{e}^{-p_0/\kappa c}, x_0 \right\rangle = -\frac{\mathrm{i}a}{\kappa c} \in \mathbb{R} \tag{38}$$

and similarly, choosing the twisting function as $F(b)$ we get the TD-twisted spacetime Hopf algebra $\mathcal{X}_\kappa(\beta)$ Hermitian basis)

$$\begin{aligned}
&[x_0, x_i] = -\frac{\mathrm{i}}{\kappa c} x_i \qquad [x_i, x_j] = 0 \\
&\Delta^{F(b)}(x_0) = \Delta_\beta(x_0) = x_0 \otimes 1 + 1 \otimes x_0 + \beta_{ij} x_i \otimes x_j \\
&\Delta^{F(b)}(x_i) = \Delta_\beta(x_i) = x_i \otimes 1 + 1 \otimes x_i \\
&S^{F(b)}(x_i) = S_\beta(x_i) = -x_i \\
&S^{F(b)}(x_0) = S_\beta(x_0) = -x_0 + \beta_{ij} x_i x_j \qquad \epsilon(x_\mu) = 0
\end{aligned} \tag{39}$$

where

$$\beta_{ij} = \frac{2\mathrm{i}}{\kappa c} b_{ij} \in \mathbb{R}. \tag{40}$$

It is obvious that in the limit $\alpha, \beta \to 0$ we obtain the κ-deformed spacetime algebra given by (13), (14).

It is well known that a twisting by a symmetric function F is equivalent to nonlinear transformation of spacetime, therefore the SD-twisted coproduct (37) is given by the function $s(x_\mu)$

$$\begin{aligned}
&\Delta_\alpha(x_0) = \Delta_0(s(x_0)) \qquad s(x_0) = x_0 \\
&\Delta_\alpha(x_i) = \Delta_0(s(x_i)) \qquad s(x_i) = x_i\, \mathrm{e}^{\alpha x_0} \\
&\Delta_0(s(x_i)) = s(x_i) \otimes \mathrm{e}^{\alpha s(x_0)} + \mathrm{e}^{\alpha s(x_0)} \otimes s(x_i)
\end{aligned} \tag{41}$$

and analogously, the TD-twisted coproduct (39) can be rewritten using function $t(x_\mu)$

$$\begin{aligned} &\Delta_\beta(x_0) = \Delta_0(t(x_0)) \qquad t(x_0) = x_0 + \tfrac{1}{2}\beta_{ij}x_i x_j \\ &\Delta_\beta(x_i) = \Delta_0(t(x_i)) \qquad t(x_i) = x_i \\ &\Delta_0(t(x_0)) = t(x_0)\otimes 1 + 1\otimes t(x_0) + \beta_{ij}t(x_i)\otimes t(x_j). \end{aligned} \tag{42}$$

We see that the twisting of κ-deformed spacetime is equivalent to nonlinear transformation of the spacetime variables.

3.2. Phase space as cross-product algebra

Let us note that we have two pairs of the dual Hopf algebras $\mathcal{X}_\kappa(\alpha)\otimes\mathcal{P}_\kappa$ and $\mathcal{X}_\kappa(\beta)\otimes\mathcal{P}_\kappa$ which in the non-deformed limit $\kappa\to\infty$ turn out to be classical spacetime and momentum algebras with multiplication defined by the commutator, therefore we should get the quantum-mechanical phase space with the standard Heisenberg commutation relations.

In order to construct such a deformed phase-space algebra Π_κ isomorphic as a vector space to $\Pi_\kappa \sim \mathcal{X}_\kappa\otimes\mathcal{P}_\kappa$, one has to extend the commutation relations (6) and (37) or (39) by adding a cross-commutator between $\mathcal{X}_\kappa$ and $\mathcal{P}_\kappa$. It appears that a consistent construction of phase space Π_κ can be done using the notion of a left (right) cross-product (smash product) algebra [9]. For simplicity, we shall consider only the left cross-product algebra.

One can define a *left action* (representation) of the momentum algebra $\mathcal{P}_\kappa$ on the spacetime algebra $\mathcal{X}_\kappa$ as a linear map

$$\triangleright : \mathcal{P}_\kappa\otimes\mathcal{X}_\kappa \to \mathcal{X}_\kappa : p\otimes x \to p\triangleright x \tag{43}$$

such that

$$(p\tilde{p})\triangleright x = p\triangleright(\tilde{p}\triangleright x) \qquad 1\triangleright x = x. \tag{44}$$

We choose the following left action

$$p\triangleright x = x_{(1)}\langle p, x_{(2)}\rangle \tag{45}$$

therefore $\mathcal{X}_\kappa$ is a left $\mathcal{P}_\kappa$-module or even a left $\mathcal{P}_\kappa$-module algebra, because $\mathcal{X}_\kappa$ and $\mathcal{P}_\kappa$ are also Hopf algebras and the left action (45) satisfies

$$p\triangleright(x\tilde{x}) = (p_{(1)}\triangleright x)(p_{(2)}\triangleright\tilde{x}) \qquad p\triangleright 1 = \epsilon(p)1. \tag{46}$$

This implies that we can regard the twisted spacetime as the left κ-deformed momentum $\mathcal{P}_\kappa$-module algebra for both choices of twisting functions (35) and (36) with the following left action (45) in the case of the SD-twisted spacetime (37)

$$\begin{aligned} &p_0\triangleright x_0 = \mathrm{i} \qquad p_i\triangleright x_0 = 0 \\ &p_0\triangleright x_i = \mathrm{i}\alpha x_i \qquad p_i\triangleright x_j = -\mathrm{i}\delta_{ij}\,\mathrm{e}^{\alpha x_0} \end{aligned} \tag{47}$$

and in the case of the TD-twisted spacetime (39) we get

$$\begin{aligned} &p_0\triangleright x_0 = \mathrm{i} \qquad p_k\triangleright x_0 = -\mathrm{i}\beta_{ik}x_i \\ &p_0\triangleright x_i = 0 \qquad p_k\triangleright x_i = -\mathrm{i}\delta_{ki}. \end{aligned} \tag{48}$$

We recall the definition of a *left cross-product algebra* [9].

Let $\mathcal{P}_\kappa$ be a Hopf algebra and $\mathcal{X}_\kappa$ a left $\mathcal{P}_\kappa$-module algebra. A left *cross-product algebra* $\Pi_\kappa = \mathcal{X}_\kappa \rtimes \mathcal{P}_\kappa$ is a vector space $\mathcal{X}_\kappa\otimes\mathcal{P}_\kappa$ with the product (*left cross-product*)

$$(x\otimes p)(\tilde{x}\otimes\tilde{p}) = x(p_{(1)}\triangleright\tilde{x})\otimes p_{(2)}\tilde{p} \tag{49}$$

and the unit element $1\otimes 1$, where $x,\tilde{x}\in\mathcal{X}_\kappa$ and $p,\tilde{p}\in\mathcal{P}_\kappa$. It appears that $\Pi_\kappa = \mathcal{X}_\kappa\rtimes\mathcal{P}_\kappa$ is an associative algebra; however, it cannot be extended to the Hopf algebra [9]. This fact suggests that one can construct κ-deformed phase space Π for a many particle system in the usual way as a tensor product algebra $\Pi = \Pi_\kappa \otimes \cdots \otimes \Pi_\kappa$.

The obvious isomorphism $\mathcal{X}_\kappa \sim \mathcal{X}_\kappa \otimes 1, \mathcal{P}_\kappa \sim 1 \otimes \mathcal{P}_\kappa$ allows us to define the commutator for the whole phase space Π_κ

$$[x, p] = x \circ p - p \circ x \qquad x \circ p = x \otimes p \qquad p \circ x = (p_{(1)} \triangleright x) \otimes p_{(2)}. \tag{50}$$

Using this definition and formulae (37) and (47), we obtain the commutation relations for the linear basis of the SD-*twisted phase space* $\Pi_\kappa(\alpha) = \mathcal{X}_\kappa(\alpha) \rtimes \mathcal{P}_\kappa$

$$\begin{aligned}
&[x_0, x_i] = -\frac{\mathrm{i}}{\kappa c} x_i && [x_i, x_j] = 0 \\
&[x_i, p_0] = -\mathrm{i}\alpha x_i && [x_0, p_i] = \frac{\mathrm{i}}{\kappa c} p_i \\
&[x_0, p_0] = -\mathrm{i} && [p_\mu, p_\nu] = 0 \\
&[x_i, p_j] = \mathrm{i}\delta_{ij}\, \mathrm{e}^{\alpha x_0} + \left(1 - \mathrm{e}^{-\mathrm{i}\alpha/\kappa c}\right) x_i p_j.
\end{aligned} \tag{51}$$

Let us note that because of the exponential term in the last relation, the SD-twisted phase space can be considered only as an algebra.

In the limit $\alpha \to 0$, we obtain the standard κ-deformed phase space considered in [17], a deformed generalization of the Heisenberg algebra. It is interesting to note that one can also consider the limit $\kappa \to \infty, \alpha =$ constant (i.e. one can assume the linear dependence of the twisting parameter a on the deformation parameter κ, see (38)) of phase space $\Pi_\kappa(\alpha) \to \Pi_\infty(\alpha)$ given by the non-vanishing commutators

$$[x_0, p_0] = -\mathrm{i} \qquad [x_i, p_j] = \mathrm{i}\delta_{ij}\, \mathrm{e}^{\alpha x_0} \qquad [x_i, p_0] = -\mathrm{i}\alpha x_i \tag{52}$$

with commuting spacetime and momentum.

Similarly, from formulae (39) and (48), we obtain the commutation relations for the linear basis of the TD-*twisted phase space* $\Pi_\kappa(\beta) = \mathcal{X}_\kappa(\beta) \rtimes \mathcal{P}_\kappa$

$$\begin{aligned}
&[x_0, x_i] = -\frac{\mathrm{i}}{\kappa c} x_i && [x_i, x_j] = 0 \\
&[x_i, p_0] = 0 && [x_0, p_i] = \frac{\mathrm{i}}{\kappa c} p_i + \beta_{ji} x_j \\
&[x_0, p_0] = -\mathrm{i} && [p_\mu, p_\nu] = 0 \\
&[x_i, p_j] = \mathrm{i}\delta_{ij}.
\end{aligned} \tag{53}$$

We see that commutators are given by linear combinations of p_i, x_i, therefore one can regard these formulae as defining phase space (not an algebra) in the classical sense. Also in this case we can consider the limit $\kappa \to \infty, \beta =$ constant (see (40)) of phase space $\Pi_\kappa(\beta) \to \Pi_\infty(\beta)$ given by the non-vanishing commutators

$$[x_0, p_0] = -\mathrm{i} \qquad [x_i, p_j] = \mathrm{i}\delta_{ij} \qquad [x_0, p_i] = \mathrm{i}\beta_{ji} x_j \tag{54}$$

with commuting spacetime and momentum.

It turns out that both phase spaces $\Pi_\infty(\alpha)$ and $\Pi_\infty(\beta)$ can be realized by the standard position and momentum operators $\hat{x}_\mu, \hat{p}_\nu$ satisfying the Heisenberg commutation relations $[\hat{x}_\mu, \hat{p}_\nu] = -\mathrm{i}g_{\mu\nu}$

$$x_i = \hat{x}_i\, \mathrm{e}^{\alpha \hat{x}_0} \qquad x_0 = \hat{x}_0 \qquad p_\mu = \hat{p}_\mu \qquad \text{for} \quad \Pi_\infty(\alpha) \tag{55}$$

and assuming $\beta_{ij} = \beta\delta_{ij}$

$$x_\mu = \hat{x}_\mu \qquad p_0 = \hat{p}_0 \qquad p_i = \hat{p}_i - \beta \hat{p}_0 \hat{x}_i \qquad \text{for} \quad \Pi_\infty(\beta). \tag{56}$$

We note that both algebras $\mathcal{X}_\kappa$ and $\mathcal{P}_\kappa$ possess the Hopf structure, therefore one can also consider a left action of spacetime algebra on the momentum algebra $\triangleright : \mathcal{P}_\kappa \otimes \mathcal{X}_\kappa \to \mathcal{X}_\kappa$

formally changing the position and momentum generators $x \leftrightarrow p$. It corresponds in quantum-mechanical language to the exchange of momentum for position representation. In this case one can define a phase space as the cross-product algebra $\mathcal{P}_\kappa \rtimes \mathcal{X}_\kappa$ (see [17]) with slightly different cross-commutation relations. However, we do not consider twisting of this kind of phase space.

4. Duality for twisted D = 2 spacetime

Considering the formulae (9) we note that different choices of coproducts $\Delta(x)$ or $\Delta(p)$ provide changes in duality relations $\langle p^m q^n, x\rangle$ and $\langle p, x^k y^l\rangle$, respectively. Therefore, we can expect the modified duality relations $\langle p_i^m p_0^n, x_j\rangle$ between four-momentum p_μ in the bicrossproduct basis (4) and the SD-twisted or TD-twisted spacetime given by relations (37) or (39). We find these twisted duality relations in the case of the two-dimensional ($D = 2$) spacetime applying tensor methods to a twisted coproduct. This simplification is convenient because of tedious calculations; however, one can immediately generalize the obtained results to the four-dimensional spacetime.

4.1. SD-twisted spacetime

In the case of the two-dimensional SD-twisted spacetime relations (37) take the following form (we assume $c = 1$ in order to simplify the notation):

(i) *spacetime*

$$[x_0, x] = \langle f(p_0), x_0\rangle x \tag{57}$$

$$\Delta(x_0) = x_0 \otimes 1 + 1 \otimes x_0 \qquad \Delta(x) = x \otimes e^{\alpha x_0} + e^{\alpha x_0} \otimes x \tag{58}$$

(ii) *momentum space*

$$[p_0, p] = 0 \tag{59}$$

$$\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0 \qquad \Delta(p) = p \otimes 1 + f(p_0) \otimes p \tag{60}$$

where we denote $f(p_0) = \exp(-p_0/\kappa)$,

(iii) *duality relations* are given by (7)–(9) for the two-dimensional case ($\mu, \nu = 0, 1$).

One can also describe these duality relations in terms of a nonlinear transformed basis of $\mathcal{X}_\kappa$ (41)

$$s(x_0) = x_0 \qquad s(x_1) = s(x) = x\, e^{\alpha x_0} \tag{61}$$

and using a trivial coproduct $\Delta_0(x_\mu)$ (14)

$$\begin{aligned} &\langle p_\mu, s(x_\nu)\rangle = \mathrm{i} g_{\mu\nu} && g_{\mu\nu} = (1, -1) \\ &\langle pq, s\rangle = \langle p \otimes q, \Delta_0(s)\rangle && \langle p, ss'\rangle = \langle \Delta(p), s \otimes s'\rangle \end{aligned} \tag{62}$$

for $s, s' \in \mathcal{X}_\kappa$, $p, q \in \mathcal{P}_\kappa$.

Taking into account the coproduct relations (58) and (60) we can immediately generalize the relations (8) and obtain

$$\langle p_\mu^m, 1\rangle = \langle 1, x_\mu^m\rangle = \delta_{m0} \qquad m = 0, 1, 2, \ldots. \tag{63}$$

In order to find other duality relations, we apply the useful relation which expresses the coassociativity of coproduct (see [9])

$$\begin{aligned} \langle p_\mu^m, x_\nu^k\rangle &= \langle \Delta^{(k-1)}(p_\mu^m), x_\nu^{\otimes k}\rangle = \langle (\Delta^{(k-1)}(p_\mu))^m, x_\nu^{\otimes k}\rangle \\ &= \langle p_\mu^{\otimes m}, \Delta^{(m-1)}(x_\nu^k)\rangle = \langle p_\mu^{\otimes m}, (\Delta^{(m-1)}(x_\nu))^k\rangle \end{aligned} \tag{64}$$

where

$$\Delta^{(n)} = (I^{\otimes(n-1)} \otimes \Delta)\Delta^{(n-1)} = (\underbrace{1 \otimes \cdots \otimes 1}_{(n-1)\text{-times}} \otimes \Delta)\Delta^{(n-1)} \tag{65}$$

$$x_\nu^{\otimes k} = \underbrace{x_\nu \otimes x_\nu \otimes \cdots \otimes x_\nu}_{k\text{-times}} \qquad p_\mu^{\otimes m} = \underbrace{p_\mu \otimes p_\mu \otimes \cdots \otimes p_\mu}_{m\text{-times}}. \tag{66}$$

In particular

$$\Delta^{(m-1)}(x) = \sum_{i=1}^{m} x_i^{(m-1)} \qquad x_i^{(m-1)} = (\mathrm{e}^{\alpha x_0})^{\otimes(i-1)} \otimes x \otimes (\mathrm{e}^{\alpha x_0})^{\otimes(m-i)} \tag{67}$$

$$\Delta^{(m-1)}(x_0) = \sum_{i=1}^{m} (x_0)_i^{(m-1)} \qquad (x_0)_i^{(m-1)} = I^{\otimes(i-1)} \otimes x_0 \otimes I^{\otimes(m-i)} \tag{68}$$

$$\Delta^{(k-1)}(p) = \sum_{i=1}^{k} p_i^{(k-1)} \qquad p_i^{(k-1)} = f^{\otimes(i-1)} \otimes p \otimes I^{\otimes(k-i)} \tag{69}$$

$$\Delta^{(k-1)}(p_0) = \sum_{i=1}^{k} (p_0)_i^{(k-1)} \qquad (p_0)_i^{(k-1)} = I^{\otimes(i-1)} \otimes p_0 \otimes I^{\otimes(k-i)}. \tag{70}$$

Using the coproduct formulae (58) for spacetime variables, duality relations and relations (67), (68) we obtain

$$\langle p_0^n, x_0 \rangle = \delta_{n1} \langle p_0, x_0 \rangle \qquad \langle p^m, x_0 \rangle = 0 \tag{71}$$

$$\langle p_0^n, x \rangle = 0 \qquad \langle p^m, x \rangle = \delta_{m1} \langle p, x \rangle. \tag{72}$$

For instance

$$\begin{aligned} \langle p^m, x_0 \rangle &= \langle p^{\otimes m}, \Delta^{(m-1)}(x_0) \rangle \\ &= \sum_{i=1}^{m} \langle p \otimes \cdots \otimes p, I^{\otimes(i-1)} \otimes x_0 \otimes I^{\otimes(m-i)} \rangle \\ &= m \langle p, 1 \rangle^{m-1} \langle p, x_0 \rangle = 0. \end{aligned}$$

Analogously, using the momentum coproduct (60) and formulae (69), (70) we derive

$$\langle p_0, x_0^l \rangle = \delta_{l1} \langle p_0, x_0 \rangle \qquad \langle p, x_0^n \rangle = 0 \tag{73}$$

$$\langle p_0, x^k \rangle = 0 \qquad \langle p, x^k \rangle = \delta_{k1} \langle p, x \rangle. \tag{74}$$

The relations (71), (72) or (73), (74) can be generalized to the following form:

$$\langle p_0^n, x_0^l \rangle = n! \langle p_0, x_0 \rangle^n \delta_{nl} \qquad \langle p^m, x_0^l \rangle = \delta_{m0} \delta_{l0} \tag{75}$$

$$\langle p^m, x^k \rangle = m! \langle p, x \rangle^m \delta_{mk} \qquad \langle p_0^n, x^k \rangle = \delta_{n0} \delta_{k0}. \tag{76}$$

From (70) and the trivial form of coproduct (60), we compute for instance

$$\begin{aligned} \langle p_0^n, x_0^l \rangle &= \langle \Delta^{(l-1)}(p_0^n), x_0^{\otimes l} \rangle \\ &= \left\langle \left(\sum_{i=1}^{l} (p_0)_i^{(l-1)} \right)^n, x_0 \otimes \cdots \otimes x_0 \right\rangle \\ &= n! \langle p_0^{\otimes n}, x_0^{\otimes l} \rangle = n! \langle p_0, x_0 \rangle^l \delta_{nl}. \end{aligned}$$

We would like to stress that the duality relations (75), (76) do not depend on the SD-twisting transformation, and they have the same form in both κ-deformed and non-deformed cases.

Let us derive the relation which depends on the twisting parameter. Using (75), (76) and coproduct formula (58) for $\Delta(x)$ we find

$$\begin{aligned}\left\langle p_0^n p^m, x\right\rangle &= \left\langle p_0^n \otimes p^m, \Delta(x)\right\rangle = \left\langle p_0^n \otimes p^m, x \otimes \mathrm{e}^{\alpha x_0} + \mathrm{e}^{\alpha x_0} \otimes x\right\rangle \\ &= \delta_{m1}\langle p, x\rangle\left\langle p_0^n, \mathrm{e}^{\alpha x_0}\right\rangle = \delta_{m1}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\langle p, x\rangle\left\langle p_0^n, x_0^k\right\rangle \\ &= \alpha^n \delta_{m1}\langle p, x\rangle\langle p_0, x_0\rangle^n. \end{aligned} \tag{77}$$

It is convenient to use instead of the power function p_0^n, its generating function $g(p_0) = \mathrm{e}^{\xi p_0}$. Then we obtain

$$\langle p^m\,\mathrm{e}^{\xi p_0}, x\rangle = \delta_{m1}\langle p, x\rangle\,\mathrm{e}^{\alpha\xi\langle p_0, x_0\rangle} \tag{78}$$

or using (69) we find a more general formula

$$\langle p^m\,\mathrm{e}^{\xi p_0}, x^k\rangle = k!(\langle p, x\rangle)^k\delta_{mk}\,\mathrm{e}^{k\alpha\xi\langle p_0, x_0\rangle} f^{\frac{1}{2}k(k-1)}(\alpha\langle p_0, x_0\rangle). \tag{79}$$

From (75) we can easily calculate the following duality relation:

$$\begin{aligned}\left\langle p^m\,\mathrm{e}^{\xi p_0}, x_0^l\right\rangle &= \sum_{s=0}^{l}\binom{l}{s}\left\langle p^m \otimes \mathrm{e}^{\xi p_0}, x_0^{l-s}\otimes x_0^s\right\rangle \\ &= \sum_{s=0}^{l}\binom{l}{s}\left\langle p^m, x_0^{l-s}\right\rangle\left\langle \mathrm{e}^{\xi p_0}, x_0^s\right\rangle \\ &= \delta_{m0}\left\langle \mathrm{e}^{\xi p_0}, x_0^l\right\rangle = (\xi\langle p_0, x_0\rangle)^l\delta_{m0} \end{aligned} \tag{80}$$

and finally we find

$$\begin{aligned}\left\langle p^m\,\mathrm{e}^{\xi p_0}, x^k x_0^l\right\rangle &= \left\langle \Delta(p^m\,\mathrm{e}^{\xi p_0}), x^k \otimes x_0^l\right\rangle \\ &= \sum_{s=0}^{m}\binom{m}{s}\left\langle p^{m-s} f^s(p_0)\,\mathrm{e}^{\xi p_0} \otimes p^s\,\mathrm{e}^{\xi p_0}, x^k \otimes x_0^l\right\rangle \\ &= \sum_{s=0}^{m}\binom{m}{s}\langle p^{m-s} f^s(p_0)\,\mathrm{e}^{\xi p_0}, x^k\rangle\left\langle p^s\,\mathrm{e}^{\xi p_0}, x_0^l\right\rangle \\ &= (\xi\langle p_0, x_0\rangle)^l\langle p^m\,\mathrm{e}^{\xi p_0}, x^k\rangle \\ &= k!(\langle p, x\rangle)^k\delta_{mk}(\xi\langle p_0, x_0\rangle)^l\,\mathrm{e}^{k\alpha\xi\langle p_0, x_0\rangle} f^{\frac{1}{2}k(k-1)}(\alpha\langle p_0, x_0\rangle) \end{aligned} \tag{81}$$

or equivalently (expanding the generating function in powers of p_0^n)

$$\left\langle p^m p_0^n, x^k x_0^l\right\rangle = k!n!(\langle p, x\rangle)^k(\langle p_0, x_0\rangle)^l\delta_{mk}\frac{(\mathrm{i}k\alpha)^{n-l}}{(n-l)!} f^{\frac{1}{2}k(k-1)}(\alpha\langle p_0, x_0\rangle). \tag{82}$$

In the limit $\alpha \to 0$ of the twisting parameter, we obtain the duality relations (12). Therefore, we see that the twisting map destroys some orthogonal duality relations, i.e. roughly speaking twisting changes the orthogonal basis of dual momentum and spacetime algebras to a non-orthogonal one.

Because twisted spacetime is a non-commuting algebra, one can also consider a polynomial basis with opposite ordering of space and time variables, i.e. left-time ordered polynomials. In order to find the duality relations for this case, we would like to stress that spacetime commutation relations (18) are related to the momentum coproduct in the bicrossproduct basis and do not depend on the twisting map, therefore we can use these relations to derive duality relations for the opposite ordering. Taking into account (7) and the explicit form of the function $f(p_0)$, we obtain the SD-*twisted duality relations*

$$\langle p^m \,\mathrm{e}^{\xi p_0}, x^k x_0^l\rangle = k!(-\mathrm{i})^k(\mathrm{i}\xi)^l\delta_{mk}\,\exp\left(\mathrm{i}k\alpha\left(\xi + \frac{1-k}{2\kappa}\right)\right) \tag{83}$$

$$\langle p^m \,\mathrm{e}^{\xi p_0}, x_0^l x^k\rangle = k!(-\mathrm{i})^k\delta_{mk}\,\exp\left(\mathrm{i}k\alpha\left(\xi + \frac{1-k}{2\kappa}\right)\right)\langle \mathrm{e}^{-kp_0/\kappa}, (x_0+\mathrm{i}\xi)^l\rangle \tag{84}$$

or equivalently

$$\langle p^m p_0^n, x^k x_0^l\rangle = k!\delta_{km}(-\mathrm{i})^k\,\exp\left(-\frac{\mathrm{i}\alpha}{2\kappa}k(k-1)\right)\langle (p_0+\mathrm{i}k\alpha)^n, x_0^l\rangle \tag{85}$$

and a non-vanishing duality relation only for $n \leqslant l$

$$\langle p^m p_0^n, x_0^l x^k\rangle = k!\delta_{km}(-\mathrm{i})^k\,\exp\left(-\frac{\mathrm{i}\alpha}{2\kappa}k(k-1)\right)\langle \mathrm{e}^{-mp_0/\kappa}(p_0+\mathrm{i}m\alpha)^n, x_0^l\rangle. \tag{86}$$

The duality relations (85), (86) one can also describe in terms of a transformed basis $s(x_\mu)$ (61) as follows

$$\langle p^m p_0^n, x^k x_0^l\rangle = \langle p^m \otimes p_0^n, \Delta_0(s^k(x)s^l(x_0))\rangle \tag{87}$$

$$\langle p^m p_0^n, x_0^l x^k\rangle = \langle p^m \otimes p_0^n, \Delta_0(s^l(x_0)s^k(x))\rangle. \tag{88}$$

4.2. *TD-twisted spacetime*

Similarly, in the two-dimensional case the TD-twisted spacetime (39) is given by the relations

$$\begin{aligned} &[x_0, x] = \langle f(p_0), x_0\rangle x \\ &\Delta(x_0) = x_0\otimes 1 + 1\otimes x_0 + \beta x\otimes x \qquad \Delta(x) = x\otimes 1 + 1\otimes x \end{aligned} \tag{89}$$

and additional formulae (59), (60) describing the momentum space and duality relations. Also in this case the relations (see (75), (76))

$$\langle p_0^n, x_0^l\rangle = n!\langle p_0, x_0\rangle^n\delta_{nl} \qquad \langle p^m, x^k\rangle = m!\langle p, x\rangle^m\delta_{mk} \tag{90}$$

are valid and the remaining ones are changed. It is easy to observe that the form of coproduct $\Delta(x_0)$ (89) implies a vanishing duality relation for any odd power of the momentum

$$\begin{aligned} &\langle p^m, x_0\rangle = \langle p^2, x_0\rangle\delta_{m2} = \beta\langle p, x\rangle^2\delta_{m2} \\ &\langle p^{2k+1}, x_0^l\rangle = 0 \quad \Leftrightarrow \quad \langle \sinh(\xi p), x_0^l\rangle = 0. \end{aligned} \tag{91}$$

Let us derive the duality relations for even power of the momentum. Using the generating function we obtain

$$\begin{aligned} \langle \mathrm{e}^{\xi p}, x_0^l\rangle &= \langle \cosh(\xi p), x_0^l\rangle = \langle \Delta^{(l-1)}\exp(\xi p), x_0^{\otimes l}\rangle \\ &= \langle \exp(\xi\Delta^{(l-1)}(p)), x_0^{\otimes l}\rangle = \left\langle \exp\left(\xi\sum_{i=1}^{l} p_i^{(l-1)}\right), x_0^{\otimes l}\right\rangle \\ &= \prod_{i=1}^{l}\langle \cosh\left(\xi p_i^{(l-1)}\right), x_0^{\otimes l}\rangle. \end{aligned} \tag{92}$$

The non-vanishing duality relations are implied by the square power of the momentum variable (91), therefore we can use the expansion

$$\cosh\left(\xi p_i^{(l-1)}\right) \sim I^{\otimes l} + \tfrac{1}{2}\xi^2\left(p_i^{(l-1)}\right)^2 \tag{93}$$

and we get

$$\left\langle \cosh(\xi p), x_0^l \right\rangle = \prod_{i=1}^{l} \left\langle \left(I^{\otimes l} + \frac{1}{2}\xi^2 \left(p_i^{(l-1)}\right)^2 \right), x_0^{\otimes l} \right\rangle$$

$$= \delta_{l0} + \sum_{k=1}^{l} \frac{1}{2^k} \xi^{2k} D_k^l \tag{94}$$

where

$$D_k^l \equiv D_k^l(\kappa, \beta) = \sum_{i_1 \neq i_2 \neq \cdots i_{k-1}}^{l-1} \left\langle \left(p_{i_1}^{(l-1)}\right)^2 \left(p_{i_2}^{(l-1)}\right)^2 \cdots \left(p_l^{(l-1)}\right)^2, x_0^{\otimes l} \right\rangle \tag{95}$$

satisfying

$$D_k^l(0,0) = D_k^l(\kappa, 0) = 0 \tag{96}$$

and comparing the appropriate left and right terms in (94) we find non-vanishing relations for $m \leqslant l$

$$\left\langle p^{2m}, x_0^l \right\rangle = \frac{(2m)!}{2^m} D_m^l(\kappa, \beta). \tag{97}$$

Now, we can derive a more general formula

$$\left\langle e^{\xi p_0} p^m, x_0^l \right\rangle = \langle e^{\xi p_0} \otimes p^m, \Delta^l(x_0) \rangle = \left\langle e^{\xi p_0} \otimes p^m, \Delta_0^l(x_0) \right\rangle$$

$$= \left\langle e^{\xi p_0} \otimes p^m, (x_0 \otimes 1 + 1 \otimes x_0)^l \right\rangle$$

$$= \sum_{s=0}^{l} \binom{l}{s} \left\langle e^{\xi p_0} \otimes p^m, x_0^{l-s} \otimes x_0^s \right\rangle$$

$$= \sum_{s=0}^{l} \binom{l}{s} \left\langle e^{\xi p_0}, x_0^{l-s} \right\rangle \left\langle p^m, x_0^s \right\rangle \tag{98}$$

or expanding we get non-vanishing duality relations for $n \leqslant l$

$$\left\langle p_0^n p^m, x_0^l \right\rangle = \frac{l!}{(l-n)!} \langle p_0, x_0 \rangle^n \left\langle p^m, x_0^{l-n} \right\rangle. \tag{99}$$

Finally, using this relation and (90) and coproduct formula (60), we derive the general duality relations for the TD-*twisted spacetime*

$$\left\langle p^m p_0^n, x^k x_0^l \right\rangle = \left\langle \Delta^m(p) \Delta^n(p_0), x^k \otimes x_0^l \right\rangle$$

$$= \frac{m!n! \langle p, x \rangle^k \langle p_0, x_0 \rangle^n}{(m-k)!(l-n)!} \left\langle p^{m-k}, x_0^{l-n} \right\rangle \tag{100}$$

or equivalently we get non-vanishing duality relations for $m - k = 2s, s \leqslant l - n\, (s = 0, 1, 2, \ldots)$

$$\left\langle p^m p_0^n, x^k x_0^l \right\rangle = \left\langle \Delta^m(p) \Delta^n(p_0), x^k \otimes x_0^l \right\rangle$$

$$= \frac{m!n! \langle p, x \rangle^k \langle p_0, x_0 \rangle^n}{(m-k)!(l-n)!} \frac{(2s)!}{2^s} D_s^{l-n}(\kappa, \beta). \tag{101}$$

In the limit $\beta \to 0$ using (96) we obtain the duality relation (12). Therefore we see that the TD-twisting appears as an additional term to the conventional duality relations (12). One can also easily find the duality relations for the opposite spacetime ordering using formula (18) but they have rather complicated form.

Also in this case one can describe the duality relations (100) in terms of the nonlinearly transformed basis (42)

$$\left\langle p^m p_0^n, x^k x_0^l \right\rangle = \left\langle p^m \otimes p_0^n, \Delta_0(t^k(x) t^l(x_0)) \right\rangle. \tag{102}$$

4.3. Spacetime and phase space; beyond the κ-deformed framework

One can give up the assumption of a commutative dual momentum space (required by κ-deformed symmetry) and consider a noncommuting four-momentum algebra dual to the spacetime algebra $\mathcal{X}_\kappa$ defined by (13), (14). In such a case we deal with a non-symmetric spacetime coproduct. As an example we shall discuss the spacetime algebra obtained by the Jordanian twist [19] (see also [20]) of the two-dimensional spacetime $\mathcal{X}_\kappa$. First we note that the relations (13), (14) for the case $D = 2$ describe the Lie algebra $B(2)$ isomorphic to a Borel subalgebra of $sl(2)$; therefore, we can apply the Jordanian twisting function (with ξ as a twisting parameter) in the form [20]

$$F_J(\kappa, \xi) = \exp(\mathrm{i}\kappa c x_0 \otimes \sigma_\xi(x)) = \exp(\mathrm{i}\kappa c x_0 \otimes \ln(1 + \xi x)) \tag{103}$$

to the trivial coproduct $\Delta_0(x_\mu)$ ($\mu = 0, 1, x_1 = x$)

$$\Delta_J(x_\mu) = F_J(\kappa, \xi)\Delta_0(x_\mu)F_J^{-1}(\kappa, \xi). \tag{104}$$

This coproduct defines the following two-dimensional Jordanian twisted spacetime $\mathcal{X}_\kappa^J(\xi)$

$$[x_0, x] = -\frac{\mathrm{i}}{\kappa c}x \qquad \epsilon(x_\mu) = 0 \tag{105}$$

$$\Delta_J(x_0) = x_0 \otimes \mathrm{e}^{-\sigma_\xi(x)} + 1 \otimes x_0 = \Delta_0(x_0) - \xi x_0 \otimes x(1 + \xi x)^{-1} \tag{106}$$

$$\Delta_J(x) = x \otimes 1 + \mathrm{e}^{\sigma_\xi(x)} \otimes x = \Delta_0(x) + \xi x \otimes x \tag{107}$$

$$S_J(x_0) = -x_0\,\mathrm{e}^{\sigma_\xi(x)} = -x_0(1 + \xi x) \tag{108}$$

$$S_J(x) = -x\,\mathrm{e}^{-\sigma_\xi(x)} = -x(1 + \xi x)^{-1}. \tag{109}$$

From the two-dimensional version of duality relations (7)–(12) one can find a dual momentum algebra $\mathcal{P}_\kappa^J(\xi)$. Because there is an isomorphism between the Hopf algebra $\mathcal{X}_\kappa^J(\xi)$ and its dual $\mathcal{P}_\kappa^J(\xi)$ [19] given by $\sigma_\xi(x) \to p_0/\kappa c,\ x_0 \to p/\kappa c\xi$, therefore we can find the defining relations ($\mu = 0, 1, p_1 = p$)

$$[p_0, p] = \mathrm{i}\kappa c\xi\left(1 - \mathrm{e}^{-p_0/\kappa c}\right) \qquad \epsilon(p_\mu) = 0 \tag{110}$$

$$\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0 \qquad S(p_0) = -p_0 \tag{111}$$

$$\Delta(p) = p \otimes 1 + \mathrm{e}^{-p_0/\kappa c} \otimes p \qquad S(p) = -\mathrm{e}^{p_0/\kappa c}p. \tag{112}$$

However, the form of the coproduct is the same as for the κ-deformed momentum algebra (60), the momentum algebra becomes noncommuting.

We can extend this pair of dual algebras to the Jordanian phase-space algebra $\Pi_\kappa^J(\xi)$ by the left cross-product construction $\Pi_\kappa^J(\xi) = \mathcal{X}_\kappa^J(\xi) \rtimes \mathcal{P}_\kappa^J(\xi)$ (49) and we find the following commutation relations of the linear basis

$$[x_0, x] = -\frac{\mathrm{i}}{\kappa c}x \qquad [p_0, p] = \mathrm{i}\kappa c\xi\left(1 - \mathrm{e}^{-p_0/\kappa c}\right) \tag{113}$$

$$[x, p_0] = 0 \qquad [x_0, p] = \mathrm{i}\left(\frac{p_0}{\kappa c} - \xi x_0\right) \tag{114}$$

$$[x_0, p_0] = -\mathrm{i} \qquad [x, p] = \mathrm{i}(1 + \xi x). \tag{115}$$

In the limit $\xi \to 0$ we obtain the standard κ-deformed phase space [17].

5. Final remarks

It is worthwhile noting that one can describe the duality relations between spacetime and momentum algebras in terms of a linearly transformed momentum basis. This possibility is similar to the description using the functions $s(x_\mu)$, $t(x_\mu)$. We consider the case of SD-twisting in the two-dimensional case. First we note that the duality relations (85) allow us to define a linear (because of the bilinear form $\langle\, ,\rangle$) transformation Φ_α in the momentum algebra $\mathcal{P}_\kappa$ corresponding to the twist operation in the spacetime $\mathcal{X}_\kappa(\alpha)$ as follows

$$\left\langle p^m p_0^n, x^k(\alpha)x_0^l(\alpha)\right\rangle = \left\langle \Phi_\alpha\left(p^m p_0^n\right), x^k x_0^l\right\rangle = \left\langle f_{mn}(\alpha), x^k x_0^l\right\rangle \tag{116}$$

where $(x(\alpha), x_0(\alpha))$ are spacetime variables (57), (58) generating the twisted algebra $\mathcal{X}_\kappa(\alpha)$ and $x, x_0 \in \mathcal{X}_\kappa$ (see (13), (14)) or in explicit form

$$\Phi_\alpha\left(p^m p_0^n\right) = f_{mn}(\alpha) = (p_0 + \mathrm{i}m\alpha)^n p^m \exp\left(-\frac{\mathrm{i}\alpha}{2\kappa} m(m-1)\right). \tag{117}$$

In particular

$$\Phi_\alpha(p^m) = f_{m0}(\alpha) = p^m \exp\left(-\frac{\mathrm{i}\alpha}{2\kappa} m(m-1)\right) \tag{118}$$

$$\Phi_\alpha\left(p_0^n\right) = f_{0n}(\alpha) = p_0^n \tag{119}$$

$$\Phi_\alpha(1) = 1 \tag{120}$$

therefore, the action of Φ_α on the momentum space (generated linearly by p_0, p) is trivial because $p_0 = f_{01}(\alpha)$, $p = f_{10}(\alpha)$ and their coproduct is given by (6). The action of Φ_α on the momentum algebra basis $p^m p_0^n$ allows us to extend this transformation onto an arbitrary polynomial function of momentum $\phi = \phi_{mn} p^m p_0^n$ in a natural way, by the replacement $p^m p_0^n \to f_{mn}(\alpha)$. Thus, one can consider the dual pair $(\mathcal{X}_\kappa(\alpha), \mathcal{P}_\kappa)$ of the twisted spacetime algebra and the κ-deformed momentum algebra or equivalently the pair of algebras $(\mathcal{X}_\kappa, \mathcal{P}_\kappa(\alpha))$ of κ-deformed spacetime and Φ_α-transformed momentum algebra with the same duality relations (116) in both cases. The essential difference in both dual constructions lies in their coalgebra sectors.

Our construction of the SD-twisted phase space algebra $\Pi_\kappa(\alpha)$ (51) relies on the notion of the cross-algebra where the multiplication (49) depends on both coproducts (58) and (60). Therefore, for the second pair of algebras $(\mathcal{X}_\kappa, \mathcal{P}_\kappa(\alpha))$ we obtain a different phase-space algebra although both pairs are equivalent as far as the duality relations are concerned. In the derivation of phase-space commutation relations we use only the first-order duality relations (for instance in the definition of the left action (47)); therefore, in the case of the pair $(\mathcal{X}_\kappa, \mathcal{P}_\kappa(\alpha))$ we obtain a two-dimensional version of commutation relations (51) for $\alpha = 0$, i.e. the standard κ-deformed phase space. One can obtain the same conclusions by considering the TD-twisting spacetime.

Therefore, one can find the linear transformation of the momentum algebra which corresponds to the twist operation in the spacetime algebra, however, this construction, does not provide the twisted phase space.

In section 3 we described two possible twistings (in space and time directions) of the spacetime algebra and derived corresponding phase spaces.

The duality relations obtained in section 4 for the two-dimensional spacetime and momentum algebras one can immediately extend to the four-dimensional case. They describe explicitly the effect of the twisting operation in the spacetime algebra.

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